HOMEWORK 6 SOLUTIONS

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Problem 9.4

Let $f : X \to Y$ and $g : Y \to Z$ be injections, and let $a, b \in X$. Suppose that $g \circ f(a) = g \circ f(b)$. Since g is injective, we must have $f(a) = f(b)$. Since f is injective we must have $a = b$. Therefore $g \circ f$ is injective.

Problem 9.5

Let f and g be bijections. Then f and g are both injections, so by problem 9.4 we know $g \circ f$ is injective. We'll show $g \circ f$ is also surjective. Let $c \in Z$ be arbitrary. Since g is a bijection it is also a surjection. Thus, there exists $b \in Y$ such that $g(b) = c$. Since f is a bijection it is also a surjection, so there exists $a \in X$ such that $f(a) = b$. Putting it all together we have $g \circ f(a) = g(f(a)) = g(b) = c$. Therefore $g \circ f$ is surjective, and hence a bijection. Now we check that that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$. For any $x \in X$ we have

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1} \circ (g^{-1} \circ g) \circ f(x) = f^{-1}(I_Y(f(x)) = f^{-1} \circ f(x) = I_X(x) = x$$

and for any $z \in Z$ we have

$$(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g \circ (f \circ f^{-1}) \circ g^{-1}(z) = g(I_Y(g^{-1}(z)) = g \circ g^{-1}(z) = I_Z(z) = z.$$

Since $f^{-1} \circ g^{-1}$ is both a left and right inverse of $g \circ f$, we conclude that it is the unique inverse.

Problem 16

(ii) This function is a bijection, and hence injective and surjective. Its inverse is $f_2^{-1}(y) = y^{1/3}$.

(iii) This function is surjective but not injective since $f(0) = f(1) = 0$.

(vi) This function is a bijection, and hence injective and surjective. Its inverse is

$$f_6^{-1}(y) = \begin{cases} \sqrt{y} \text{ if } y \geq 0 \\ -\sqrt{-y} \text{ if } y \leq 0 \end{cases}$$

Problem 18

Let $f : X \to Y$ and $g : Y \to Z$ be surjections. To show $g \circ f$ is surjective, let $c \in Z$ be arbitrary. Since g is surjective, there exists $b \in Y$ such that $g(b) = c$. Since f is surjective there exists $a \in X$ such that $f(a) = b$. Then we have $g \circ f(a) = g(f(a)) = g(b) = c$, so $g \circ f$ is surjective.

Problem 19

First suppose there exists a function $g : Y \to X$ such that $f \circ g = I_Y$. Let $b \in Y$ be arbitrary. Then $g(b) \in X$ and we have $f(g(b)) = (f \circ g)(b) = I_Y(b) = b$, so f is surjective.

Now suppose f is surjective. We'll define a function $g : Y \to X$ as follows. Fix $b \in Y$. Since f is surjective, there exists $a \in X$ such that $f(a) = b$. Note: there may be many, even infinitely many, such possibilities. Just pick one such a, and define $g(b) = a$. In other words, for any $b \in Y$ we require that $g(b) \in \{x \in X | f(x) = b\}$, and surjectivity of f guarantees that this set is nonempty. Then for any $b \in Y$ we have $f \circ g(b) = f(g(b)) = b$, so $f \circ g = I_Y$.

Problem 20

(i) Let $b \in \overrightarrow{f}$. Then there exists $a \in A_1$ such that $f(a) = b$. Since $A_1 \subseteq A_2$, this implies $a \in A_2$, which means $b = f(a) \in \overrightarrow{f}(A_2)$, so $\overrightarrow{f}(A_1) \subseteq \overrightarrow{f}(A_2)$. To see that the converse is not true, consider $f : \mathbb{R} \to \mathbb{R}$ define by $f(x) = x^2$. Let $A_1 = \{-1, 1\}$ and $A_2 = \{0, 1\}$. Then $\overrightarrow{f}(A_1) = \{1\} \subseteq \{0, 1\} = \overrightarrow{f}(A_2)$, but A_1 is not a subset of A_2. In general, the converse fails if and only if f is not injective.

(ii) Let $b \in \overrightarrow{f}(A_1 \cap A_2)$. Then there exists $a \in A_1 \cap A_2$ such that $f(a) = b$. Since $a \in A_1$, $b = f(a) \in \overrightarrow{f}(A_1)$. Since $a \in A_2$, $b = f(a) \in \overrightarrow{f}(A_2)$. Thus, $b \in \overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2)$. To see that equality doesn't always hold, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $A_1 = \{-1\}$ and $A_2 = \{1\}$. Then $\overrightarrow{f}(A_1 \cap A_2) = \overrightarrow{f}(\varnothing) = \varnothing$ but $\overrightarrow{f}(A_1) \cap \overrightarrow{f}(A_2) = \{1\} \cap \{1\} = \{1\}$.

(iii) First we'll show $\overrightarrow{f}(A_1 \cup A_2) \subseteq \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$. Let $b \in \overrightarrow{f}(A_1 \cup A_2)$. Then there exists $a \in A_1 \cup A_2$ such that $f(a) = b$. If $a \in A_1$, then $b = f(a) \in \overrightarrow{f}(A_1)$ so $b \in \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$. If $a \in A_2$ then $b = f(a) \in \overrightarrow{f}(A_2)$ so $b \in \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$.

Now we'll show $\overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2) \subseteq \overrightarrow{f}(A_1 \cup A_2)$. Suppose $b \in \overrightarrow{f}(A_1) \cup \overrightarrow{f}(A_2)$. If $b \in \overrightarrow{f}(A_1)$ then there exists $a \in A_1$ such that $b = f(a)$. Since $a \in A_1$, we have $a \in A_1 \cup A_2$ so $b = f(a) \in \overrightarrow{f}(A_1 \cup A_2)$. On the other hand, if $b \in \overrightarrow{f}(A_2)$ then there exists $a \in A_2$ such that $b = f(a)$. Since $a \in A_2$, we have $a \in A_1 \cup A_2$ so $b = f(a) \in \overrightarrow{f}(A_1 \cup A_2)$.